

Solutions to Maths workbook – 1 | Trigonometric Identities and Equations

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Daily Tutorial Sheet - 2

9.(C) $x^2 - 2x \sec \theta + 1 = 0 \Rightarrow x = \sec \theta \pm \tan \theta$

As $\alpha_1 > \beta_1$ and $\theta \in IV^{th}$ quadrant so, $\alpha_1 = \sec \theta - \tan \theta$ and $\beta_1 = \sec \theta + \tan \theta$

Also, $x^2 + 2x \tan \theta - 1 = 0 \Rightarrow x = -\tan \theta \pm \sec \theta$.

As $\alpha_2 > \beta_2$ & $\theta \in IV^{th}$ quadrant so, $\alpha_2 = -\tan \theta + \sec \theta$ and $\beta_2 = -\tan \theta - \sec \theta$

$\therefore \alpha_1 + \beta_2 = \sec \theta - \tan \theta - \tan \theta - \sec \theta = -2 \tan \theta$

10.(C) Let $T_k = \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right) \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)} \Rightarrow T_k = \frac{1}{\sin\left(\frac{\pi}{6}\right) \left[\frac{\sin\left(\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) - \left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\right)}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right) \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)} \right]}$

$$\Rightarrow T_k = 2 \left[\frac{\cos\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)} - \frac{\cos\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}{\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)} \right] \Rightarrow T_k = 2 \left[\cot\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right) - \cot\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) \right]$$

$$\Rightarrow \sum_{k=1}^{13} T_k = 2 \left[\cot\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{4} + \frac{13\pi}{6}\right) \right] \Rightarrow 2 \left[1 - \cot\frac{5\pi}{12} \right] = 2 \left[1 - (2 - \sqrt{3}) \right] = 2(\sqrt{3} - 1)$$

11.(AB) $\frac{\sin^4 x}{2} + \frac{(1 - \sin^2 x)^2}{3} = \frac{1}{5} \Rightarrow \frac{\sin^4 x}{2} + \frac{1 + \sin^4 x - 2 \sin^2 x}{3} = \frac{1}{5}$

$$\Rightarrow 5(5 \sin^4 x + 2 - 4 \sin^2 x) = 6 \Rightarrow 25 \sin^4 x - 20 \sin^2 x + 4 = 0$$

$$\Rightarrow (5 \sin^2 x - 2)^2 = 0 \Rightarrow \sin^2 x = \frac{2}{5} \Rightarrow \cos^2 x = \frac{3}{5}$$

So, $\tan^2 x = \frac{2}{3}$ and $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

12.(B) Let $\cot \theta = t$

$\theta \in \left(0, \frac{\pi}{4}\right) \Rightarrow t \in (1, \infty)$

$\tan \theta = \frac{1}{t} \in (0, 1)$

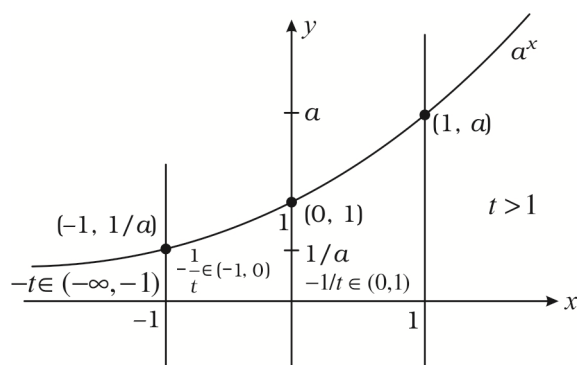
Consider any constant $a > 1$

Then: $a^t \in (a, \infty)$ when $t \in (1, \infty)$

$a^{1/t} \in (1, a)$ when $t \in (1, \infty)$ because $\frac{1}{t} \in (0, 1)$

$a^{-t} = \frac{1}{a^t} \in \left(0, \frac{1}{a}\right), \quad a^{-1/t} = \frac{1}{a^{1/t}} \in \left(\frac{1}{a}, 1\right)$

One can also ascertain this from the graph of a^x



Hence, clearly, $a^t > a^{1/t} > a^{-1/t} > a^{-t}$ (1)

For every $a > 1$ and $t > 1$

Since $\cot \theta = t > 1$

$$t_4 = t^t, \quad t_3 = t^{1/t}, \quad t_2 = \left(\frac{1}{t}\right)^t = t^{-t}, \quad t_1 = \left(\frac{1}{t}\right)^{1/t} = t^{-1/t}$$

Using (1), we get $t_4 > t_3 > t_1 > t_2$

13.(D) $\cos(\alpha - \beta) = 1 \Rightarrow \alpha - \beta = 2k\pi$

Now, $-\pi < \alpha, \beta < \pi \Rightarrow -2\pi < \alpha - \beta < 2\pi \therefore \alpha - \beta = 0 \Rightarrow \alpha = \beta$

Now $\cos(\alpha + \beta) = \frac{1}{e} \Rightarrow \cos 2\alpha = \frac{1}{e} \Rightarrow 2\alpha = 2n\pi \pm \cos^{-1} \frac{1}{e} \Rightarrow \alpha = n\pi \pm \frac{1}{2} \cos^{-1} \left(\frac{1}{e}\right)$

$\therefore$ 4 values of α are possible : $-\pm \frac{1}{2} \cos^{-1} \left(\frac{1}{e}\right), \pi - \frac{1}{2} \cos^{-1} \frac{1}{e}$ and $-\pi + \frac{1}{2} \cos^{-1} \left(\frac{1}{e}\right)$

$\alpha = \beta \Rightarrow$ There are 4 ordered pairs of (α, β) .

14.(D) $5(\sec^2 x - \cos^2 x - 1) = 2(2 \cos^2 x - 1) + 9$

Let $\cos^2 x = t \Rightarrow 5\left(\frac{1}{t} - t - 1\right) = 2(2t - 1) + 9 \Rightarrow t = \frac{1}{3}, \frac{-5}{3} \Rightarrow \cos^2 x = \frac{1}{3}, \frac{-5}{3}$ (reject)

$\therefore \cos^2 x = \frac{1}{3} \Rightarrow \cos 2x = 2 \cos^2 x - 1 = \frac{-7}{9}$

15.(D) $f_4(x) = \frac{1}{4}(\sin^4 x + \cos^4 x) = \frac{1}{4}(1 - 2 \sin^2 x \cos^2 x)$ and $f_6(x) = \frac{1}{6}(\sin^6 x + \cos^6 x) = \frac{1}{6}(1 - 3 \sin^2 x \cos^2 x)$

$\Rightarrow f_4(x) - f_6(x) = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

16.(B) $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} = \frac{\tan^2 A}{\tan A - 1} - \frac{1}{\tan A(\tan A - 1)}$

$\Rightarrow \frac{\tan^3 A - 1}{\tan A(\tan A - 1)} = \frac{\tan^2 A + 1 + \tan A}{\tan A} \Rightarrow \frac{\sec^2 A}{\sin A / \cos A} + 1 \Rightarrow \sec A \operatorname{cosec} A + 1$

17.(B) Squaring and adding given equations we have:

$25 + 24 \sin(P + Q) = 37 \Rightarrow \sin(P + Q) = \frac{1}{2}$ So, $P + Q = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

Case-I: $P + Q = \frac{\pi}{6} \Rightarrow P < \frac{\pi}{6}, Q < \frac{\pi}{6} \Rightarrow 3 \sin P + 4 \cos Q < \frac{3}{2} + 4 < 5.5 < 6$

$$\therefore P + Q = \frac{\pi}{6} \text{ is not possible} \quad \Rightarrow \quad P + Q = \frac{5\pi}{6} \text{ is only possibility} \quad \Rightarrow \quad R = \frac{\pi}{6}$$

$$\mathbf{18.(D)} \quad A = \sin^2 x + \cos^4 x = \cos^4 x - \cos^2 x + 1 \Rightarrow A = \left(\cos^2 x - \frac{1}{2} \right)^2 + \frac{3}{4}$$

$$\text{Now, } 0 \leq \cos^2 x \leq 1 \Rightarrow \frac{-1}{2} \leq \cos^2 x - \frac{1}{2} \leq \frac{1}{2} \Rightarrow 0 \leq \left(\cos^2 x - \frac{1}{2} \right)^2 \leq \frac{1}{4}$$

$$\Rightarrow 0 + \frac{3}{4} \leq A \leq \frac{1}{4} + \frac{3}{4} \Rightarrow \frac{3}{4} \leq A \leq 1$$

$$\mathbf{19.(B)} \quad \text{For given condition: } - 0 \leq \alpha, \beta \leq \frac{\pi}{4}$$

$$\cos(\alpha + \beta) = \frac{4}{5} \Rightarrow \tan(\alpha + \beta) = \frac{3}{4} \quad \text{and} \quad \sin(\alpha - \beta) = \frac{5}{13} \Rightarrow \tan(\alpha - \beta) = \frac{5}{12}$$

$$\therefore \tan 2\alpha = \tan((\alpha + \beta) + (\alpha - \beta)) = \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta)\tan(\alpha - \beta)} \Rightarrow \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \times \frac{5}{12}} \Rightarrow \frac{56}{33}$$

$$\mathbf{20.(C)} \quad \left[(\cos \beta \cos \gamma + \sin \beta \sin \gamma) + (\cos \gamma \cos \alpha + \sin \alpha \sin \gamma) + \cos \alpha \cos \beta + \sin \alpha \sin \beta \right] = -\frac{3}{2}$$

$$1 + 1 + 1 + 2[\sum \cos \alpha \cos \beta + \sum \sin \alpha \sin \beta] = 0$$

$$\cos^2 \alpha + \sin^2 \alpha + \cos^2 \beta + \sin^2 \beta + \cos^2 \gamma + \sin^2 \gamma + 2[\sum \cos \alpha \cos \beta + \sum \sin \alpha \sin \beta] = 0$$

$$(\cos \alpha + \cos \beta + \cos \gamma)^2 + (\sin \alpha + \sin \beta + \sin \gamma)^2 = 0$$

$$\Rightarrow \cos \alpha + \cos \beta + \cos \gamma = 0 \quad \text{and} \quad \sin \alpha + \sin \beta + \sin \gamma = 0$$